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INTEGRATION BY AUXILIARY INTEGRALS.

BY WERNER A. STILLE, PH. D., HIGHLAND, ILL.

[Continued from page 114.]

We have seen that all (or nearly all) the integrals contained in the larger collections may be found with far less labor than by the methods of the books when simply employing the ordinary primitive integrals as auxiliaries, in our present sense of the word. When the integration of a function, $dv = f(x).dx$, is required we take for auxiliary such primitive function $du = \varphi(x).dx$ as may appear most suitable, then we form the expression $dv \div du$ in terms of u and integrate, thus finding v in terms of u. Now, our new primitive integrals were devised for the purpose of enlarging the range of integrable functions and to facilitate the integration of functions already otherwise integrable. This latter remark applies also to that class of functions known as "binomial" which we have thus far only slightly touched upon and which by means of our method become integrable without the laborious process of rationalization and decomposition into partial fractions.

When a binomial is proposed for integration it is convenient to employ different auxiliaries, varying with the nature of the constant quantities occurring in the proposed function. For illustration, let it be required to integrate

$$dv = \frac{(2.x^5 - 1)^{1/2}}{x^{11}} dx.$$

Here it is convenient to take the auxiliary

$$du = \frac{dx}{x\sqrt{(2x^5 - 1)}}; \therefore u = \tfrac{1}{5}\text{arc sin}\left(\frac{x^5 - 1}{x^5}\right);$$

$$\therefore \frac{1}{x^5} = 1 - \sin 5u; \cos 5u = \frac{1}{x^5}\sqrt{(2x^5 - 1)};$$

$$\therefore \frac{dv}{du} = \frac{2x^5 - 1}{x^{10}} = \cos^2 5u;$$

$$v = \int \cos^2 5u\, du$$

the integration of which needs no further remark. Again, let it be required to integrate

$$dv = \frac{dx}{x^3(x^4 - 1)^{\frac{5}{4}}}.$$

Here we take for auxiliary

$$du = \frac{dx}{x\sqrt{(x^4 - 1)}}; \therefore u = \tfrac{1}{2}\text{arc tg}\sqrt{(x^4 - 1)}; \therefore x = \sec 2u,$$

$$\therefore v = \int \frac{\cos^3 2u}{\sin^2 2u} du,$$

which is integrable without any difficulty.

If we wished to extend the ordinary reference-tables of integrals, so as to give new forms, one mode of so doing would be to write the same integrals contained in the tables in terms of our own variables by substituting our variables for x. One example of this kind will suffice. Take the well known integral

$$v = \int \frac{du}{a + b.\cos u} = \frac{1}{\sqrt{(a^2-b^2)}} \operatorname{arc\,cos} \left(\frac{b+a\cos u}{a+b\cos u}\right). \quad (a^2>b^2.)$$

Now, putting $u = -\operatorname{arc\,sin}\sqrt{[(1-x^2)\div(1+x^2)]}$,

$$du = \sqrt{2}.\frac{dx}{(1+x^2)\sqrt{(1-x^2)}}; \quad \cos u = x\sqrt{\frac{2}{1+x^2}};$$

$$dv = \frac{\sqrt{2}.dx}{(1+x^2)\sqrt{(1-x^2)}\left[a+bx\sqrt{\dfrac{2}{1+x^2}}\right]};$$

$$\therefore \int \frac{\sqrt{2}.dx}{[a\sqrt{(1+x^2)}+bx\sqrt{2}]\sqrt{(1-x^4)}} = \frac{1}{\sqrt{(a^2-b^2)}}$$
$$\times \operatorname{arc\,cos}\left[\frac{b\sqrt{(1+x^2)}+ax\sqrt{2}}{a\sqrt{(1+x^2)}+bx\sqrt{2}}\right].$$

A great variety of new forms of integrals is obtained by the substitution of such forms as $k \operatorname{tg} \varphi$, $1 + k \operatorname{tg} \varphi$, $\sqrt{(1+k \operatorname{tg} \varphi)}$, &c. for the variable x in those integrals belonging to the goniometric types. Thus take the ordinary primitive form

$$y = \int \frac{dx}{\sqrt{(1-x^2)}} = \operatorname{are\,sin} x; \quad \therefore x = \sin y,$$

and make $x = k \operatorname{tg} \varphi$, then

$$y = \int \frac{k.d\varphi}{\cos^2\varphi\sqrt{(1-k^2.\operatorname{tg}^2\varphi)}} = \int \frac{2k.d\varphi}{\sqrt{[(1-k^2)+2\cos 2\varphi+(k^2+1).\cos^2 2\varphi]}}.$$

Putting $\cos 2\varphi = t$; $\therefore 2d\varphi = -[dt\div\sqrt{(1-t^2)}]$, we have

$$y = -\int \frac{k.dt}{\sqrt{(1+t^2)}\sqrt{[(1-k^2)+2t+(k^2+1)t^2]}} = \operatorname{arc\,sin}(k \operatorname{tg} \varphi)$$
$$= \operatorname{arc\,sin} \left\{ k\sqrt{[(1-t)\div(1+t)]} \right\}.$$

This formula may, of course, be used as a primitive or auxiliary, and it is easy to develop by means of it a great number of new integrals. The simplest method again would be to express in terms of t any integral known in terms of y.

Treating in the same way any of our primitive forms we arrive at the corresponding new forms which in their turn may be employed as auxiliaries. As an example of the kind, take

$$dy = \frac{dx}{(2+x)\sqrt{(1+x)}};\ \therefore y = 2\,\text{arc tg}\sqrt{(1+x)};\ \therefore 2+x = \frac{1}{\cos^2\frac{1}{2}y};$$

$$\therefore \cos\tfrac{1}{2}y = \frac{1}{\sqrt{(2+x)}};\ \therefore \sin\tfrac{1}{2}y = \sqrt{\frac{1+x}{2+x}}.$$

Putting $x = k\,\text{tg}\,\varphi$,

$$dy = \frac{k\,d\varphi}{\cos^2\varphi(2+k\,\text{tg}\,\varphi)\sqrt{(1+k\,\text{tg}\,\varphi)}} = \frac{k\,d\varphi}{(2\cos\varphi+k\sin\varphi)\sqrt{(\cos^2+k\sin\varphi\cos\varphi)}}$$

and this, putting $\cos\varphi = t$, gives

$$\int\frac{-k\,dt}{[2t+k\sqrt{(1-t^2)}]\sqrt{(1-t^2)}\,[t^2+kt\sqrt{(1-t^2)}]^{\frac{1}{2}}} = 2\,\text{arc tg}\left(k\frac{\sqrt{(1-t^2)}}{t}+1\right)^{\frac{1}{2}}$$

which equation may be used as an auxiliary.

Another fruitful method of finding new integrals is the combination of our elementary forms, so as to express the one in terms of the other. As an example take

$$v = \int\frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = -\frac{1}{\sqrt{2}}\,\text{arc sin}\sqrt{\frac{1-x^2}{1+x^2}}$$

and compare it with

$$u = \int\frac{2x\,dx}{\sqrt{(1-x^2)}\sqrt{(1+x^2)}} = \text{arc sin}\,x^2;\ \therefore x^2 = \sin u;$$

$$\therefore \frac{dv}{du} = \frac{1}{2x\sqrt{(1+x^2)}} = \frac{1}{2\sqrt{(\sin u)}\sqrt{(1+\sin u)}};$$

$$\therefore\ v = \int\frac{du}{2\sqrt{(\sin u)}\sqrt{(1+\sin u)}} = -\sqrt{2}.\text{arc sin}\left(\frac{1-\sin u}{1+\sin u}\right)^{\frac{1}{2}}$$

and comparing this again with

$$u = \sqrt{2}\int\frac{dt}{(1+t^2)\sqrt{(1-t^2)}} = -\,\text{arc sin}\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{2}}$$

$$\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{2}} = -\sin u;\ \therefore \sqrt{(\sin u)} = i\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{4}},$$ we find

$$v = \frac{1}{\sqrt{2}}\int\frac{dt}{(1+t^2)\sqrt{(1-t^2)}.i\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{4}}\left[1-\left(\frac{1-t^2}{1+t^2}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}}$$

$$v = \frac{-i}{\sqrt{2}}\int\frac{dt}{(1-t^2)^{\frac{3}{4}}(1+t^2)^{\frac{1}{2}}[\sqrt{(1+t^2)}-\sqrt{(1-t^2)}]^{\frac{1}{2}}}$$

$$= -\sqrt{2}.\text{arc sin}\left[\frac{(1+t^2)^{\frac{1}{2}}-(1-t^2)^{\frac{1}{2}}}{(1+t^2)^{\frac{1}{2}}+(1-t^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}}.$$

A great number of new integrals are found by the differentiation of such equations as

$$y = \text{arc sin}\,(ax^{\frac{1}{2}}+bx^{\frac{3}{2}}),$$
$$y = \text{arc sin}\,(ax^{-\frac{1}{2}}+bx^{\frac{3}{2}}),$$
$$y = \text{arc sin}\,(ax^{-\frac{1}{2}}+bx^{\frac{1}{2}}+cx^{\frac{3}{2}}),$$

&c. &c. &c.

and then using the integrals so found as auxiliaries.